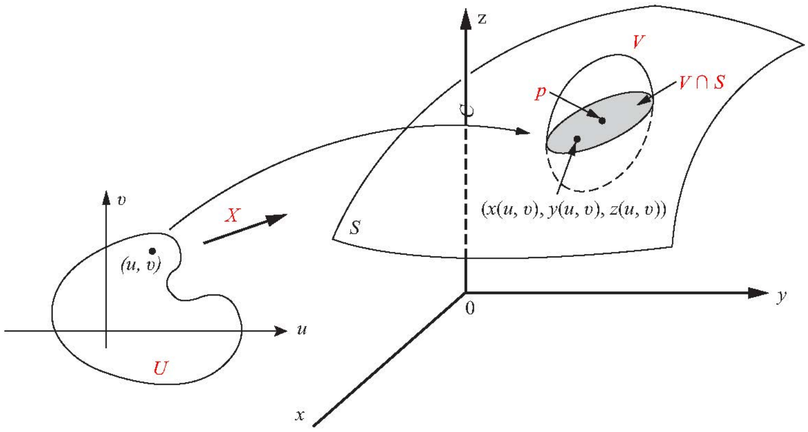
Regular Surfaces

Definition 1. A subset $S \subset \mathbb{R}^3$ is a regular surface if, for each $p \in S$, there exists an open neighborhood V in $\mathbb{R}^3$, an open set $U \subset \mathbb{R}^2$ and a map

$$X : U \to V \cap S$$



such that

(1) X is smooth, meaning that if we write

$$X(u,v) = (x(u,v), y(u,v), z(u,v)), \quad (u,v) \in U,$$

then the real-valued functions $x(u,v)$, $y(u,v)$ and $z(u,v)$ have continuous partial derivatives of all orders in U.

(2) X is a homeomorphism, meaning that it is a one-to-one correspondence between the points of U and $V \cap S$ which is continuous in both directions; that is, X^{-1} is the restriction of a continuous map $F : W \subset \mathbb{R}^3 \to \mathbb{R}^2$ defined on an open set W containing $V \cap S$.

(3) (The regularity condition) For each $q = (u,v) \in U$, the linear map

$$dX_q = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix} : \mathbb{R}^2 \to \mathbb{R}^3,$$

called the differential of X at q, is one-to-one.

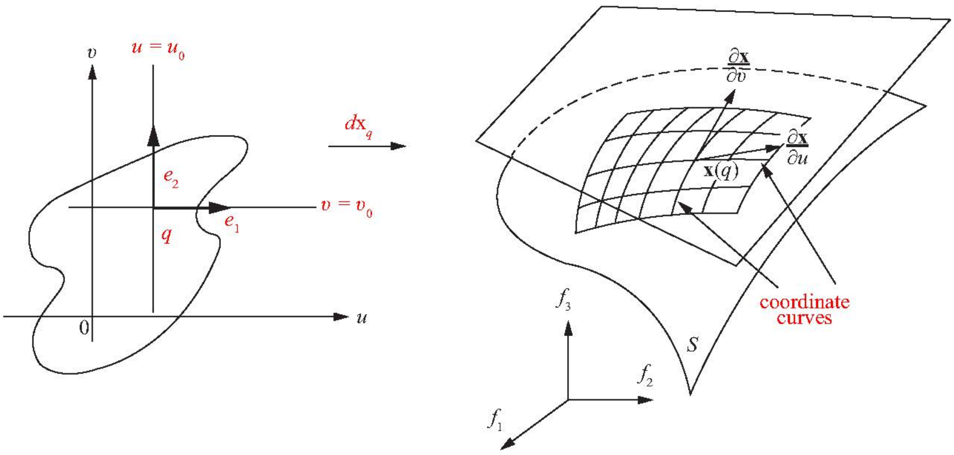
The mapping $X : U \to V \cap S$ is called a parametrization or a system of local coordinates for the surface S in the coordinate neighborhood $V \cap S$ of p.

Remarks Note that a surface is defined as a subset S of $\mathbb{R}^3$, not as a map as in the curve case. This is achieved by covering S with the traces of parameterization which satisfy the three conditions. Also note that

- Condition (1) is natural if we need to do differential calculus on S.
- Condition (2) has the purpose of preventing self-intersection in regular surfaces. It is also essential to prove that certain objects defined in terms of a parameterization do not depend on this parameterization but only on S itself.
- To give condition 3 a more familiar form, let us compute the matrix of the linear map dX_q in the canonical bases $e_1 = (1, 0)$, $e_2 = (0, 1)$ of $\mathbb{R}^2$ with coordinates (u, v) and $f_1 = (1, 0, 0)$, $f_2 = (0, 1, 0)$, $f_3 = (0, 0, 1)$ of $\mathbb{R}^3$, with coordinates (x, y, z).

  Let $q = (u_0, v_0)$. The vector e_1 is tangent to the curve $u \to (u, v_0)$ whose image under X is the curve

$$u \to (x(u, v_0), y(u, v_0), z(u, v_0)).$$



This image curve (called the coordinate curve $v = v_0$) lies on S and has at $X(q)$ the tangent vector

$$\left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right) = \frac{\partial X}{\partial u},$$

where the derivatives are computed at (u_0, v_0) and a vector is indicated by its components in the basis $\{f_1, f_2, f_3\}$. By the definition of differential

$$dX_q(e_1) = \frac{d}{du}X(u, v_0)|_{u=u_0} = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right) = \frac{\partial X}{\partial u}.$$

Similarly, using the coordinate curve $u = u_0$ (image by X of the curve $v \to (u_0, v)$), we obtain

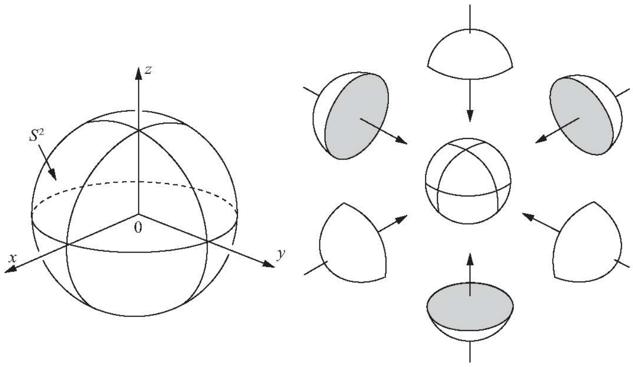
$$dX_q(e_2) = \frac{d}{dv}X(u_0, v)|_{v=v_0} = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right) = \frac{\partial X}{\partial v}.$$

Condition (3) may now be expressed by requiring the two column vectors of this matrix to be linearly independent; or, equivalently, that the vector product $\frac{\partial X}{\partial u} \wedge \frac{\partial X}{\partial v} \neq 0$; or, in still another way, that one of the minors of order 2 of the matrix of dX_q, that is, one of the Jacobian determinants

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}, \quad \frac{\partial(y, z)}{\partial(u, v)}, \quad \frac{\partial(x, z)}{\partial(u, v)},$$

is different from zero at q. So, Condition (3) will gurantee the existence of a tangent plane at all points of S.

Example Show that the unit sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3; x^2 + y^2 + z^2 = 1\}$ is a regular surface.



- Let $U = \{(x, y) \in \mathbb{R}^2; x^2 + y^2 < 1\}$, $\mathbb{R}^2 = \{(x, y, z) \in \mathbb{R}^3; z = 0\}$ and the maps X_1, $X_2 : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be defined by

$$X_1(x, y) = (x, y, +\sqrt{1 - x^2 - y^2}), \quad X_2(x, y) = (x, y, \sqrt{1 - x^2 - y^2}), \quad (x, y) \in U.$$

- Let $U = \{(x, z) \in \mathbb{R}^2; x^2 + z^2 < 1\}$, $\mathbb{R}^2 = \{(x, y, z) \in \mathbb{R}^3; y = 0\}$ and the maps X_3, $X_4 : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be defined by

$$X_3(x, z) = (x, +\sqrt{1 - x^2 - z^2}, z), \quad X_4(x, y) = (x, -\sqrt{1 - x^2 - z^2}, z), \quad (x, z) \in U.$$

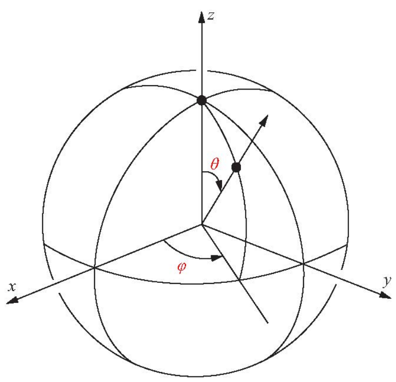
- Let $U = \{(y, z) \in \mathbb{R}^2; y^2 + z^2 < 1\}$, $\mathbb{R}^2 = \{(x, y, z) \in \mathbb{R}^3; x = 0\}$ and the maps X_5, $X_6 : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be defined by

$$X_5(y, z) = (+\sqrt{1 - x^2 - z^2}, y, z), \quad X_6(y, z) = (-\sqrt{1 - x^2 - z^2}, y, z), \quad (y, z) \in U.$$

Since $\{X_i : U \to S^2 \mid 1 \leq i \leq 6\}$ are parametrizations covering S^2 completely, S^2 is a regular surface.

For most applications, it is convenient to relate parametrizations to the geographical coordinates on S^2. Let $V = \{(\theta, \phi); 0 < \theta < \pi, 0 < \phi < 2\pi\}$ and let $X : V \to \mathbb{R}^3$ be given by

$$X(\theta, \phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta).$$



Clearly, $X(V) \subset S^2$ and X is a parametrization of S^2, where θ is usually called the colatitude (the complement of the latitude) and ϕ the longitude. Note that $X(V)$ only omits a semicircle of S^2 (including the two poles) and that S^2 can be covered with the coordinate neighborhoods of two parametrizations of this type.

Example shows that deciding whether a given subset of $\mathbb{R}^3$ is a regular surface directly from the definition may be quite tiresome. Before going into further examples, we shall present two propositions which will simplify this task. Proposition 1 shows the relation which exists between the definition of a regular surface and the graph of a function $z = f(x, y)$. Proposition 2 uses the inverse function theorem and relates the definition of a regular surface with the subsets of the form $f(x, y, z) =$ constant.

Proposition 1. If $f : U \to \mathbb{R}$ is a differentiable function in an open set U of $\mathbb{R}^2$, then the graph of f over U, that is, the subset of $\mathbb{R}^3$ given by

$$\begin{aligned} S &= \text{the graph of a differentiable function } f \text{ on an open subset } U \subset \mathbb{R}^2 \\ &= \{(x, y, f(x, y)) \mid (x, y) \in U \subset \mathbb{R}^2\} \end{aligned}$$

is a regular surface.

Proof It suffices to show that the map $X : U \to \mathbb{R}^3$ given by

$$X(u, v) = (u, v, f(u, v)) \quad \text{for } (u, v) \in U$$

is a parametrization of the graph whose coordinate neighborhood covers every point of the graph.

Definition 2. Given a differentiable map $F : U \subset \mathbb{R}^n \to \mathbb{R}^m$ defined in an open set U of $\mathbb{R}^n$, we say that $p \in U$ is a critical point of F if the differential $dF_p : \mathbb{R}^n \to \mathbb{R}^m$ is not a surjective (or onto) mapping. The image $F(p) \in \mathbb{R}^m$ of a critical point is called a critical value of F. A point of $\mathbb{R}^m$ which is not a critical value is called a regular value of F.

Remark Recall that

Implicit Function Theorem

Let $U \subseteq \mathbb{R}^{n+m} \equiv \mathbb{R}^n \times \mathbb{R}^m$ be an open set,
$f = (f_1, \dots, f_m) : U \to \mathbb{R}^m$ belong to Class $C^1(U)$,
$(a, b) = (a_1, \dots, a_n, b_1, \dots, b_m) \in U$ be a point at which $f(a, b) = 0 \in \mathbb{R}^m$,
and the $m \times m$ matrix $D_y f|_{(a,b)} = \left[\dfrac{\partial f_i}{\partial y_j}(a, b)\right]_{1 \le i,j \le m}$ be invertible.

Then there exists

an open neighborhood A of a in $\mathbb{R}^n$,
an open neighborhood B of b in $\mathbb{R}^m$,
and a unique $g : A \to B$ belonging to Class $C^1(A)$

such that

$$b = g(a) \quad \text{and} \quad f(x, g(x)) = 0 \in \mathbb{R}^m \quad \text{for all } x \in A$$

and thus

$$\begin{aligned} f^{-1}(0) \cap A \times B &= \{(x, y) \in A \times B \subset U \subseteq \mathbb{R}^n \times \mathbb{R}^m \mid f(x, y) = 0 \in \mathbb{R}^m\} \\ &= \{(x, g(x)) \mid x \in A\} \\ &= \text{the graph of } g \text{ over } A. \end{aligned}$$

Example (2-2 Exercises #17) Prove that

(a) The inverse image of a regular value of a differentiable function

$$f : U \subset \mathbb{R}^2 \to \mathbb{R}$$

is a regular plane curve. Give an example of such a curve which is not connected.

Proof Let r be a regular value of f. For each

$$p = (p_1, p_2) \in f^{-1}(r) \cap U = \{(x, y) \in U \subset \mathbb{R}^2 \mid f(x, y) = r\} \subset \mathbb{R}^2,$$

since $df_p = (f_x(p), f_y(p)) : \mathbb{R}^2 \to \mathbb{R}$ is surjective, $(f_x(p), f_y(p)) \neq (0, 0)$. By interchanging x and y if necessary, we may assume that $f_y(p) \neq 0$, thus, by the Implicit Function Theorem, there exists an open set

$$I_1 \times I_2 = (a_1, b_1) \times (a_2, b_2) \quad \text{containg } p = (p_1, p_2)$$

and a unique continuously differentiable function $g : I_1 \to I_2$ such that

$$f(x, g(x)) = r \text{ for all } x \in I_1 \implies f^{-1}(r) \cap I_1 \times I_2 = \{(x, g(x)) \mid x \in I_1\} \subset \mathbb{R}^2$$

is a regular plane over $I_1 = (a_1, b_1)$ since g is differentiable and the tangent vector $(1, g'(x)) \neq (0, 0)$ at each $(x, g(x))$, $x \in I_1$. Since p is an arbitrary point in $f^{-1}(r) \cap U$, $f^{-1}(r)$ is a regular plane curve.

Example Let $U = \{(x, y) \mid x \neq 0\}$ and $f(x, y) = xy$ for $(x, y) \in U$. Then $f^{-1}(1) = \{(x, \frac{1}{x}) \mid x > 0\} \cup \{(x, \frac{1}{x}) \mid x < 0\}$= disjoint union of two regular plane curves.

(b) The inverse image of a regular value of a differentiable map

$$F = (F_1, F_2) : U \subset \mathbb{R}^3 \to \mathbb{R}^2$$

is a regular curve in $\mathbb{R}^3$.

Proof Let $r = (r_1, r_2)$ be a regular value of $F = (F_1, F_2)$. For each

$$p = (p_1, p_2, p_3) \in F^{-1}(r) \cap U = \{(x, y, z) \in U \subset \mathbb{R}^3 \mid F(x, y, z) = r\} \subset \mathbb{R}^3,$$

since

$$dF_p = \begin{pmatrix} (F_1)_x(p) & (F_1)_y(p) & (F_1)_z(p) \\ (F_2)_x(p) & (F_2)_y(p) & (F_2)_z(p) \end{pmatrix} : \mathbb{R}^3 \to \mathbb{R}^2 \quad \text{is surjective,}$$

the matrix $\begin{pmatrix} (F_1)_x(p) & (F_1)_y(p) & (F_1)_z(p) \\ (F_2)_x(p) & (F_2)_y(p) & (F_2)_z(p) \end{pmatrix}$ is of rank 2. By interchanging x, y or z if necessary, we may assume that

$$\begin{pmatrix} (F_1)_y(p) & (F_1)_z(p) \\ (F_2)_y(p) & (F_2)_z(p) \end{pmatrix} \quad \text{is nonsingular, } (F_1)_z(p) \neq 0, \ (F_2)_y(p) \neq 0,$$

thus, by the Implicit Function Theorem, there exist an open neighborhood $I_1 \times I_2 \times I_3 = (a_1, b_1) \times (a_2, b_2) \times (a_3, b_3)$ of (p_1, p_2, p_3) and unique continuously differentiable functions $g = (g_1, g_2) : I_1 \to I_2 \times I_2$, $h : I_1 \times I_2 \to I_3$, $k : I_1 \times I_3 \to I_2$ such that

$$F(x, g(x)) = r \text{ for all } x \in I_1 \implies F^{-1}(r) \cap I_1 \times I_2 \times I_3 = \{(x, g_1(x), g_2(x)) \mid x \in I_1\} \subset \mathbb{R}^3$$

is a regular curve in $\mathbb{R}^3$

$$\begin{aligned}
& F_1(x,y,h(x,y)) = r_1 \text{ for all } (x,y) \in I_1 \times I_2 \\
\Longrightarrow\ & (F_1)^{-1}(r_1) \cap I_1 \times I_2 \times I_3 = \{(x,y,h(x,y)) \mid (x,y) \in I_1 \times I_2\} \subset \mathbb{R}^3 \\
& \text{is a graph of a differentiable function } h \text{ over } I_1 \times I_2 \\
\Longrightarrow\ & (F_1)^{-1}(r_1) \cap I_1 \times I_2 \times I_3 \text{ is a regular surface in } \mathbb{R}^3
\end{aligned}$$

$$\begin{aligned}
& F_2(x,k(x,z),z) = r_2 \text{ for all } (x,z) \in I_1 \times I_3 \\
\Longrightarrow\ & (F_2)^{-1}(r_2) \cap I_1 \times I_2 \times I_3 = \{(x,k(x,z).z) \mid (x,z) \in I_1 \times I_3\} \subset \mathbb{R}^3 \\
& \text{is a graph of a differentiable function } k \text{ over } I_1 \times I_3 \\
\Longrightarrow\ & (F_2)^{-1}(r_2) \cap I_1 \times I_2 \times I_3 \text{ is a regular surface in } \mathbb{R}^3
\end{aligned}$$

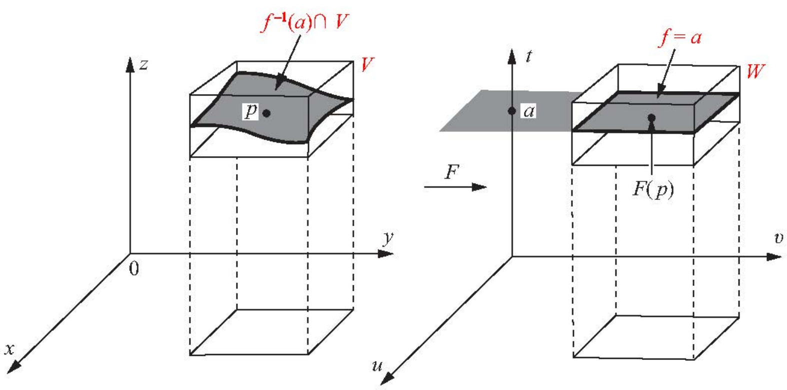
Since p is an arbitrary point in $F^{-1}(r) \cap U$, these imply that $F^{-1}(r)$ is a regular curve, $(F_1)^{-1}(r_1)$ and $(F_2)^{-1}(r_2)$ are regular surfaces in $\mathbb{R}^3$ and $F^{-1}(r) = (F_1)^{-1}(r_1) \cap (F_2)^{-1}(r_2)$

Proposition 2. If $f : U \subset \mathbb{R}^3 \to \mathbb{R}$ is a differentiable function and $a \in f(U)$ is a regular value of f, then the level set of a regular value $f^{-1}(a) = \{(x,y,z) \in U \mid f(x,y,z) = a\}$ is a regular surface in $\mathbb{R}^3$.

Proof Let $p = (x_0, y_0, z_0)$ be a point of $f^{-1}(a)$. Since a is a regular value of f, it is possible to assume, by renaming the axes if necessary, that $f_z(p) \neq 0$. We define a mapping $F : U \subset \mathbb{R}^3 \to \mathbb{R}^3$ by $F(x,y,z) = (x,y,f(x,y,z))$ for $(x,y,z) \in U$. Since

$$\det(dF_p) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ f_x(p) & f_y(p) & f_z(p) \end{vmatrix} = f_z(p) \neq 0,$$

and by the inverse funtion theorem, there exist open neighborhoods V of p and W of $F(p)$ such that $F : V \to W$ is invertible and the inverse $F^{-1} : W \to V$ is differentiable.



Let F^{-1} be defined by $F^{-1}(u,v,t) = (g_1(u,v,t), g_2(u,v,t), g_2(u,v,t))$ for $(u,v,t) \in W$. Since

$$(u,v,t) = F \circ F^{-1}(u,v,t) = F(g_1,g_2,g_3) = (g_1, g_2, f(g_1,g_2,g_3)) \quad \text{for } (u,v,t) \in W,$$

we have $g_1(u,v,t) = u$, $g_2(u,v,t) = v$ and

$$(x,y,z) = F^{-1}(u,v,t) = (u,v,g_3(u,v,t)) \quad \text{for } (u,v,t) \in W,\ (x,y,z) \in V.$$

This implies that $\pi(V) \cong \pi(W)$ and $z = g_3(u, v, a) = h(u, v)$ is a differentiable function for $(u, v) \in \pi(W)$, where $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ is the projection map defined by $\pi(u, v, t) = (u, v)$ for $(u, v, t) \in \mathbb{R}^3$. Since

$$
\begin{aligned}
& F(f^{-1}(a) \cap V) = W \cap \{(u, v, t) \mid t = a\} \\
\Longrightarrow \quad & f^{-1}(a) \cap V = F^{-1}(W \cap \{(u, v, t) \mid t = a\}) = \{(u, v, g_3(u, v, a)) \mid (u, v) \in \pi(W)\} \\
\underset{g_3(x,y,a)=h(x,y)}{\overset{\pi(W)\cong\pi(V)}{\Longrightarrow}} \quad & f^{-1}(a) \cap V = \{(x, y, h(x, y)) \mid (x, y) \in \pi(V)\},
\end{aligned}
$$

we conclude that $f^{-1}(a) \cap V$ is the graph of h over $\pi(V)$. By Prop. 1, $f^{-1}(a) \cap V$ is a coordinate neighborhood of p. Therefore, every $p \in f^{-1}(a)$ can be covered by a coordinate neighborhood, and so $f^{-1}(a)$ is a regular surface.

Example The ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

is a regular surface. In fact, it is the set $f^{-1}(0)$ where

$$f(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1$$

is a differentiable function and 0 is a regular value of f. This follows from the fact that the partial derivatives $f_x = 2x/a^2$, $f_y = 2y/b^2$, $f_z = 2z/c^2$ vanish simultaneously only at the point $(0, 0, 0)$, which does not belong to $f^{-1}(0)$. This example includes the sphere as a particular case $(a = b = c = 1)$.

Example The hyperboloid of two sheets $-x^2 - y^2 + z^2 = 1$ is a regular surface, since it is given by $S = f^{-1}(0)$, where 0 is a regular value of $f(x, y, z) = -x^2 - y^2 + z^2 - 1$. Note that the surface S is not connected; that is, given two points in two distinct sheets ($z > 0$ and $z < 0$) it is not possible to join them by a continuous curve $\alpha(t) = (x(t), y(t), z(t))$ contained in the surface; otherwise, z changes sign and, for some t_0, we have $z(t_0) = 0$, which means that $\alpha(t_0) \in S$.

Example Let $a > r > 0$, $S^1 = \{(y, z) \mid (y - a)^2 + z^2 = r^2\}$ and T be the surface, called torus, generated by rotating S^1 about z-axis. Hence the points (x, y, z) of T satisfy the equation

$$z^2 = r^2 - (\sqrt{x^2 + y^2} - a)^2.$$

Therefore, T is the inverse image of r^2 by the function

$$f(x, y, z) = z^2 + (\sqrt{x^2 + y^2} - a)^2.$$

Note that f is differentiable for $(x, y) \neq (0, 0)$, and r^2 is a regular value of f. It follows that the torus T is a regular surface.

Proposition 3. Let $S \subset \mathbb{R}^3$ be a regular surface and $p \in S$. Then there exists an open neighborhood V of p in S such that V is the graph of a differentiable function which has one of the following three forms:

$$z = f(x, y), \quad y = g(x, z), \quad x = h(y, z).$$

That is, regular surface is locally a graph of a differentiable function.

Proof Let $X : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a parametrization of S at p, and write $X(u, v) = (x(u, v), y(u, v), z(u, v))$, $(u, v) \in U$. By condition 3 of Def. 1, one of the Jacobian determinants

$$\frac{\partial(x, y)}{\partial(u, v)} \quad \frac{\partial(y, z)}{\partial(u, v)} \quad \frac{\partial(z, x)}{\partial(u, v)}$$

is not zero at $X^{-1}(p) = q$.

Suppose first that $\dfrac{\partial(x,y)}{\partial(u,v)}(q) \neq 0$, and consider the map $\pi \circ X : U \to \mathbb{R}^2$, where $\pi(x,y,z) = (x,y)$ for $(x,y,z) \in \mathbb{R}^3$, defined by

$$\pi \circ X(u,v) = (x(u,v), y(u,v)), \quad \text{for } (u,v) \in U.$$

Since $\det d(\pi \circ X)(q) = \dfrac{\partial(x,y)}{\partial(u,v)}(q) \neq 0$, by the inverse function theorem, there exist open neighborhoods V_1 of q, V_2 of $\pi \circ X(q)$ such that

- $\pi \circ X : V_1 \to V_2$ is one-to-one, onto and has a differentiable inverse $(\pi \circ X)^{-1} : V_2 \to V_1$ defined by
$$(\pi \circ X)^{-1}(x,y) = (u(x,y), v(x,y)) \quad \text{for } (x,y) \in V_2.$$

It follows that

- the projection map $\pi : X(V_1) = V \subset S \to V_2$ is one-to-one on V,

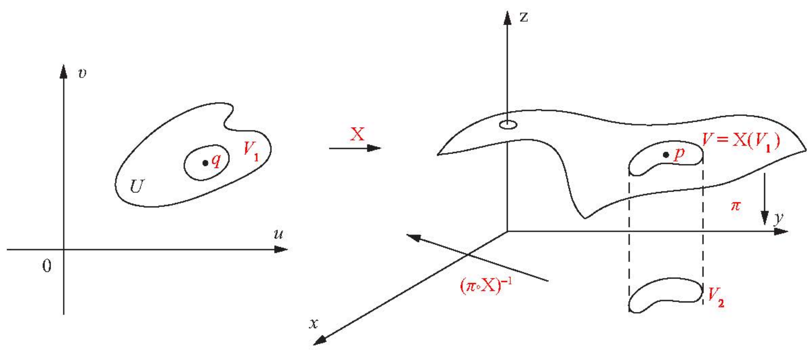
$\Longrightarrow$ V is a graph of $z = z \circ (\pi \circ X)^{-1}(x,y)$ on V_2.

In fact, since X is a homeomorphism, $V = X(V_1) = (X^{-1})^{-1}(V_1)$ is an open neighborhood of p in S and since

- $z = z(u,v)$ is differentiable for $(u,v) \in V_1$,
- $(u(x,y), v(x,y)) = (\pi \circ X)^{-1}(x,y)$ is differentiable for $(x,y) \in V_2$,

$\Longrightarrow$ $z = z(u(x,y), v(x,y)) = z \circ (\pi \circ X)^{-1}(x,y)$ is differentiable in $\in V_2$,

and



$$\begin{aligned} V &= \{X(u,v) = (x(u,v), y(u,v), z(u,v)) \mid (u,v) \in V_1\} \\ &= \{(x,y,z) \mid z = z(u(x,y), v(x,y)) = z \circ (\pi \circ X)^{-1}(x,y),\ (x,y) \in V_2\}, \end{aligned}$$

V is the graph of the differentiable function $z = z(u(x,y), v(x,y)) = f(x,y)$ over V_2, and this settles the first case.

The remaining cases can be treated in the same way, yielding $x = h(y,z)$ and $y = g(x,z)$.

Proposition 4. Let $p \in S$ be a point of a regular surface S and let $X : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a map with $p \in X(U) \subset S$ such that conditions (1) and (3) of Def. 1 hold. Assume that X is one-to-one. Then X^{-1} is continuous.

Proof Let $X : U \subset \mathbb{R}^2 \to S \subset \mathbb{R}^3$ be defined by

$$X(u,v) = (x(u,v), y(u,v), z(u,v)), \quad \text{for } (u,v) \in U,$$

and let $p = X(q)$ for some $q \in U$.

By conditions 1 and 3, we can assume, by interchanging the coordinate axis if necessary, that

$$\frac{\partial(x,y)}{\partial(u,v)}(q) \neq 0.$$

By the inverse function theorem, there exist open neighborhoods V_1 of q, V_2 of $\pi \circ X(q)$ such that $\pi \circ X$ maps V_1 diffeomorphically onto V_2, where $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ is the projection defined by $\pi(x,y,z) = (x,y)$ for $(x,y,z) \in \mathbb{R}^3$. It follows that $(\pi \circ X)^{-1} : V_2 \to V_1$ and $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ are continuous maps.

Assume now that X is one-to-one. Then $X : V_1 \to V = X(V_1) \subset S$ has an inverse $X^{-1} : V \to V_1$ and, since $X^{-1} = (\pi \circ X)^{-1} \circ \pi : V \to V_1$ is a composition of continuous maps, X^{-1} is continuous.

Examples

1. The one-sheetd cone C, given by $z = +\sqrt{x^2+y^2}$, $(x,y) \in \mathbb{R}^2$, is not a regular surface.

   If C were a regular surface, it would be, in a neighborhood of $(0,0,0) \in C$, the graph of a differentiable function having one of three forms: $y = h(x,z)$, $x = g(y,z)$, $z = f(x,y)$.

   The two first forms can be discarded by the simple fact that the projections of C over the xz and yz planes are not one-to-one. The last form would have to agree, in a neighborhood of $(0,0,0)$, with $z = +\sqrt{x^2+y^2}$. Since $z = +\sqrt{x^2+y^2}$ is not differentiable at $(0,0)$, this is impossible.

2. Let $a > r > 0$, $S^1 = \{(y,z) \mid (y-a)^2 + z^2 = r^2\}$ and T be the torus generated by rotating S^1 about z-axis. Then

   $$X(u,v) = ((r\cos u + a)\cos v, (r\cos u + a)\sin v, r\sin u) \quad \text{where } 0 < u < 2\pi,\ 0 < v < 2\pi$$

   is a parametrization for the torus T.

   Condition 1 of Def. 1 is easily checked, and condition 3 reduces to a straightforward computation, which is left as an exercise. Since we know that T is a regular surface, condition 2 is equivalent, by Prop. 4, to the fact that X is one-to-one.

   To prove that X is one-to-one, we first observe that $\sin u = z/r$ also, if $\sqrt{x^2+y^2} \leq a$, then $\pi/2 \leq u \leq 3\pi/2$, and if $\sqrt{x^2+y^2} \geq a$, then either $0 < u \leq \pi/2$ or $3\pi/2 \leq u < 2\pi$. Thus, given (x,y,z), this determines u, $0 < u < 2\pi$, uniquely. By knowing u, x, and y we find $\cos v$ and $\sin v$. This determines v uniquely, $0 < v < 2\pi$. Thus, X is one-to-one.

   It is easy to see that the torus can be covered by three such coordinate neighborhoods.

Change of Parameters; Differentiable Functions on Surface

Proposition 1 (Change of Parameters). Let p be a point of a regular surface S, and let $X : U \subset \mathbb{R}^2 \to S$, $Y : V \subset \mathbb{R}^2 \to S$ be two parametrizations of S such that $p \in X(U) \cap Y(V) = W$. Then the "change of coordinates" $h = X^{-1} \circ Y : Y^{-1}(W) \to X^{-1}(W)$ is a diffeomorphism; that is, h is differentiable and has a differentiable inverse h^{-1}.

In other words, if X and Y are given by

$$\begin{aligned} X(u,v) &= (x(u,v), y(u,v), z(u,v)), \quad (u,v) \in U \\ Y(\xi,\eta) &= (x(\xi,\eta), y(\xi,\eta), z(\xi,\eta)), \quad (\xi,\eta) \in V \end{aligned}$$

then the change of coordinates h, given by

$$u = u(\xi,\eta), \quad v = v(\xi,\eta), \quad (\xi,\eta) \in Y^{-1}(W),$$

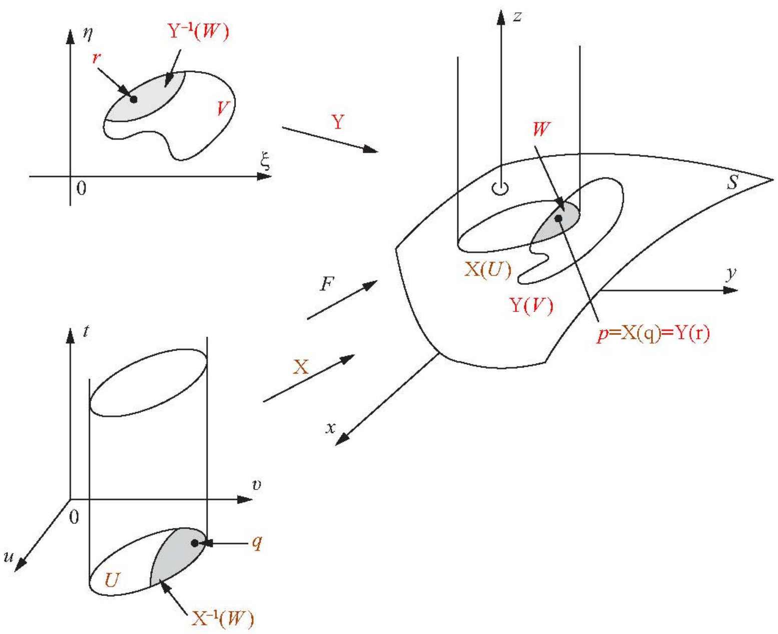
has the property that the functions u and v have continuous partial derivatives of all orders, and the map h can be inverted, yielding

$$\xi = \xi(u,v), \quad \eta = \eta(u,v) \quad (u,v) \in X^{-1}(W),$$

where the functions ξ and η also have partial derivatives of all orders. Since

$$\frac{\partial(u,v)}{\partial(\xi,\eta)} \cdot \frac{\partial(\xi,\eta)}{\partial(u,v)} = 1,$$

this implies that the Jacobian determinants of both h and h^{-1} are nonzero everywhere.



Proof $h = X^{-1} \circ Y$ is a homeomorphism, since it is composed of homeomorphisms. It is not possible to conclude, by an analogous argument, that h is differentiable, since X^{-1} is defined in an open subset of S, and we do not yet know what is meant by a differentiable function on S.

We proceed in the following way. Let $r \in Y^{-1}(W)$ and set $q = h(r) \in X^{-1}(W)$.
Since $X(u,v) = (x(u,v), y(u,v), z(u,v))$ is a parametrization, we can assume, by renaming the axes if necessary, that

$$\frac{\partial(x,y)}{\partial(u,v)}(q) \neq 0.$$

We extend X to a map $F : U \times \mathbb{R} \to \mathbb{R}^3$ defined by

$$F(u,v,t) = (x(u,v), y(u,v), z(u,v) + t), \quad (u,v) \in U, \ t \in \mathbb{R}.$$

Geometrically, F maps a vertical cylinder C over U into a "vertical cylinder" over $X(U)$ by mapping each section of C with height t into the surface $X(u,v)+te_3$, where e_3 is the unit vector of the z axis.

It is clear that F is differentiable and that the restriction $F|_{U\times\{0\}} = X$. Since

$$\det dF_q = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & 0 \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & 0 \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & 1 \end{vmatrix} = \frac{\partial(x,y)}{\partial(u,v)}(q) \neq 0,$$

we can apply the inverse function theorem to find an open neighborhood M of $p = X(q)$ in $\mathbb{R}^3$ such that F^{-1} exists and is differentiable in M.

By the continuity of Y, there exists an open neighborhood $N = Y^{-1}(M \cap Y(V)) \subset V$ of r in V such that $Y(N) \subset M \cap S$. Since $X^{-1}|_{Y(N)} = F^{-1}|_{Y(N)}$, and restricted to N,

$$\begin{aligned} & h|_N = X^{-1} \circ Y|_N = F^{-1} \circ Y|_N \text{ is a composition of differentiable maps} \\ \implies & h \text{ is differentiable at } r \in Y^{-1}(W). \\ \implies & \text{Since } r \text{ is arbitrary, } h \text{ is differentiable on } Y^{-1}(W). \end{aligned}$$

Exactly the same argument can be applied to show that the map h^{-1} is differentiable, and so h is a diffeomorphism.

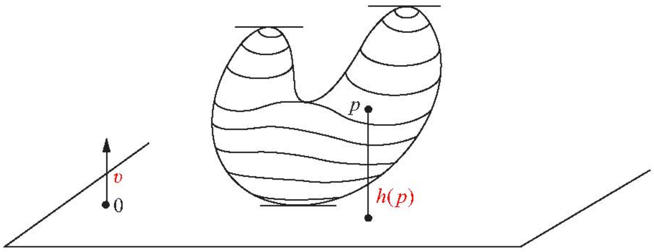
Definition Let $f : V \subset S \to \mathbb{R}$ be a function defined in an open subset V of a regular surface S. Then f is said to be differentiable at $p \in V$ if, for some parametrization $X : U \subset \mathbb{R}^2 \to S$ with $p \in X(U) \subset V$, the composition $f \circ X : U \subset \mathbb{R}^2 \to \mathbb{R}$ is differentiable at $X^{-1}(p)$. f is differentiable in V if it is differentiable at all points of V.

It follows immediately from the last proposition that the definition given does not depend on the choice of the parametrization X. In fact, if $Y : V \subset \mathbb{R}^2 \to S$ is another parametrization with $p \in Y(V)$, and if $h = X^{-1} \circ Y$, then $f \circ Y = f \circ X \circ h$ is also differentiable, whence the asserted independence.

Remark 1 We shall frequently make the notational abuse of indicating f and $f \circ X$ by the same symbol $f(u,v)$, and say that $f(u,v)$ is the expression of f in the system of coordinates X. This is equivalent to identifying $X(U)$ with U and thinking of (u,v), indifferently, as a point of U and as a point of $X(U)$ with coordinates (u,v). From now on, abuses of language of this type will be used without further comment.

Example 1 Let S be a regular surface and $V \subset \mathbb{R}^3$ be an open set such that $S \subset V$. Let $f : V \subset \mathbb{R}^3 \to \mathbb{R}$ be a differentiable function. Then the restriction of f to S is a differentiable function on S. In fact, for any $p \in S$ and any parametrization $X : U \subset \mathbb{R}^2 \to S$ in p, the function $f \circ X : U \to \mathbb{R}$ is differentiable. In particular, the following are differentiable functions:

1. The height function relative to a unit vector $v \in \mathbb{R}^3$, $h : S \to \mathbb{R}$, given by $h(p) = p \cdot v$, $p \in S$, where the dot denotes the usual inner product in $\mathbb{R}^3$. $h(p)$ is the height of $p \in S$ relative to a plane normal to v and passing through the origin of $\mathbb{R}^3$.
2. The square of the distance from a fixed point $p_0 \in \mathbb{R}^3$, $f(p) = |p - p_0|^2$, $p \in S$. The need for taking the square comes from the fact that the distance $|p - p_0|$ is not differentiable at $p = p_0$.



Remark 2 The proof of Prop. 1 makes essential use of the fact that the inverse of a parametrization is continuous. Since we need Prop. 1 to be able to define differentiable functions on surfaces (a vital concept), we cannot dispose of this condition in the definition of a regular surface.

The definition of differentiability can be easily extended to mappings between surfaces.

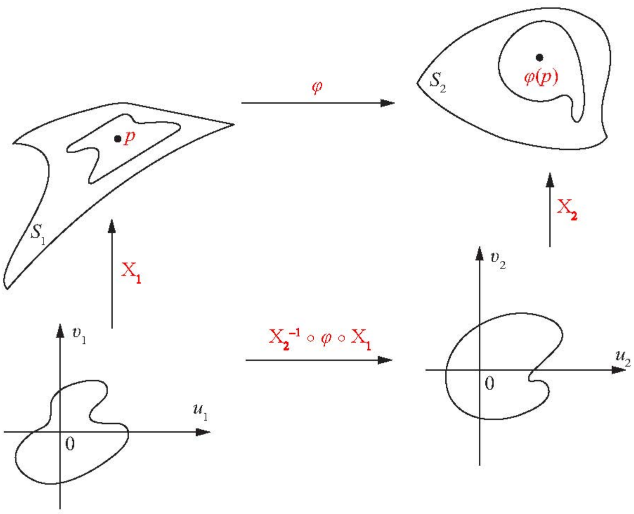
Definition A continuous map $\varphi : V_1 \subset S_1 \to S_2$ of an open set V_1 of a regular surface S_1 to a regular surface S_2 is said to be differentiable at $p \in V_1$ if, given parametrizations

$$X_1 : U_1 \subset \mathbb{R}^2 \to S_1, \quad X_2 : U_2 \subset \mathbb{R}^2 \to S_2$$

with $p \in X_1(U_1)$ and $\varphi(X_1(U_1)) \subset X_2(U_2)$, the map

$$X_2^{-1} \circ \varphi \circ X_1 : U_1 \to U_2$$

is differentiable at $q = X_1^{-1}(p)$.



In other words, φ is differentiable if when expressed in local coordinates as

$$\varphi(u_1, v_1) = (\varphi_1(u_1, v_1), \varphi_2(u_1, v_1))$$

the functions φ_1 and φ_2 have continuous partial derivatives of all orders.

Note that one can use Prop. 1 to show that this definition of differentiability of $\varphi : S_1 \to S_2$ does not depend on the choice of parametrizations.

We should mention that the natural notion of equivalence associated with differentiability is the notion of diffeomorphism.

Definition Two regular surfaces S_1 and S_2 are diffeomorphic if there exists a differentiable map $\varphi : S_1 \to S_2$ with a differentiable inverse $\varphi^{-1} : S_2 \to S_1$. Such a φ is called a diffeomorphism from S_1 to S_2.

Note that, if $\varphi : S_1 \to S_2$ is a diffeomorphism from S_1 to S_2, then $f : S_2 \to \mathbb{R}$ is differentiable on S_2 if and only if $f \circ \varphi : S_1 \to \mathbb{R}$ is differentiable on S_1, i.e. two diffeomorphic surfaces are indistinguishable from the point of view of differentiability.

Example 2 If $X : U \subset \mathbb{R}^2 \to S$ is a parametrization, then $X^{-1} : X(U) \to U \subset \mathbb{R}^2$ is differentiable.

In fact, for any $p \in X(U)$ and any parametrization $Y : V \subset \mathbb{R}^2 \to S$ in p, we have that

$$X^{-1} \circ Y : Y^{-1}(X(U) \cap Y(V)) \to X^{-1}(X(U) \cap Y(V)) \quad \text{is differentiable.}$$

This shows that U and $X(U)$ are diffeomorphic (i.e., every regular surface is locally diffeomorphic to a plane) and justifies the identification made in Remark 1.

Example 3 Let S_1 and S_2 be regular surfaces. Assume that $S_1 \subset V \subset \mathbb{R}^3$, where V is an open set of $\mathbb{R}^3$, and that $\varphi : V \to \mathbb{R}^3$ is a differentiable map such that $\varphi(S_1) \subset S_2$. Then the restriction $\varphi|_{S_1} : S_1 \to S_2$ is a differentiable map.

In fact, given $p \in S_1$ and parametrizations

$$X_1 : U_1 \subset \mathbb{R}^2 \to S_1, \quad X_2 : U_2 \subset \mathbb{R}^2 \to S_2$$

with $p \in X_1(U_1)$ and $\varphi(X_1(U_1)) \subset X_2(U_2)$, we have that the map

$$X_2^{-1} \circ \varphi \circ X_1 : U_1 \to U_2 \quad \text{is differentiable.}$$

The following are particular cases of this general example:

1. Let S be symmetric relative to the xy plane; that is, if $(x, y, z) \in S$, then also $(x, y, -z) \in S$. Then the (antipodal) map $\sigma : S \to S$, which takes $p \in S$ into its symmetrical point, is differentiable, since it is the restriction to S of the differentiable map $\sigma : \mathbb{R}^3 \to \mathbb{R}^3$ defined by
$$\sigma(x, y, z) = (x, y, z) \quad \text{for } (x, y, z) \in \mathbb{R}^3.$$
This, of course, generalizes to surfaces symmetric relative to any plane of $\mathbb{R}^3$.
2. Let $R_{z,\theta} : \mathbb{R}^3 \to \mathbb{R}^3$ be the rotation of angle θ about the z axis, and let $S \subset \mathbb{R}^3$ be a regular surface invariant by this rotation; i.e., $\{R_{z,\theta}(p) \mid p \in S\} \subseteq S$. Then the restriction $R_{z,\theta} : S \to S$ is a differentiable map.
3. Let $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ be given by
$$\varphi(x, y, z) = (xa, yb, zc), \quad \text{where } a, b \text{ and } c \text{ are nonzero real numbers.}$$
Then φ is clearly differentiable, and the restriction $\varphi|_{S^2}$ is a differentiable map from the sphere
$$S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$$
into the ellipsoid
$$\{(x, y, z) \in \mathbb{R}^3 \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\}.$$

Remark 3 Proposition 1 implies (cf. Example 2) that a parametrization $X : U \subset \mathbb{R}^2 \to S$ is a diffeomorphism of U onto $X(U)$. Actually, we can now characterize the regular surfaces as those subsets $S \subset \mathbb{R}^3$ which are locally diffeomorphic to $\mathbb{R}^2$; that is, for each point $p \in S$, there exists a neighborhood V of p in S, an open set $U \subset \mathbb{R}^2$, and a map $X : U \to V$, which is a diffeomorphism. This pretty characterization could be taken as the starting point of a treatment of surfaces (see Exercise 13).

Definition A parametrized surface $X : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ is a differentiable map X from an open set $U \subset \mathbb{R}^2$ into $\mathbb{R}^3$. The set $X(U) \subset \mathbb{R}^3$ is called the trace of X. X is regular if the differential $dX_q : \mathbb{R}^2 \to \mathbb{R}^3$ is one-to-one for all $q \in U$ (i.e., the vectors $\partial X/\partial u$, $\partial X/\partial v$ are linearly independent for all $q \in U$). A point $p \in U$ where dX_p is not one-to-one is called a singular point of X.

Observe that a parametrized surface, even when regular, may have self-intersections in its trace since $X : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ is not necessary a homeomorphism (or a global one-to-one map).

Example Let $\alpha : I \to \mathbb{R}^3$ be a nonplanar regular parametrized curve. Define

$$X(t, v) = \alpha(t) + v\alpha'(t), \quad (t, v) \in I \times \mathbb{R}.$$

X is a parametrized surface called the tangent surface of α.

Suppose that the curvature $k(t) \neq 0$, for all $t \in I$, and restrict the domain of X to

$$U = \{(t, v) \mid (t, v) \in I \times \mathbb{R};\ v \neq 0\}.$$

Since $k(t) = \dfrac{|\alpha''(t) \wedge \alpha'(t)|}{|\alpha'(t)|^3} \neq 0$, $\dfrac{\partial X}{\partial t} = \alpha'(t) + v\alpha''(t)$ and $\dfrac{\partial X}{\partial v} = \alpha'(t)$, we have

$$\frac{\partial X}{\partial t} \wedge \frac{\partial X}{\partial v} = v\alpha''(t) \wedge \alpha'(t) \neq 0 \quad \text{for all } (t, v) \in U.$$

It follows that the restriction $X : U \to \mathbb{R}^3$ is a regular parametrized surface, the trace of which consists of two connected pieces whose common boundary is the set $\alpha(I)$.

Proposition Let $X : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a regular parametrized surface and let $q \in U$. Then there exists a neighborhood V of q in $\mathbb{R}^2$ such that $X(V) \subset \mathbb{R}^3$ is a regular surface.

Proof Let $X : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be defined by

$$X(u, v) = (x(u, v), y(u, v), z(u, v)), \quad (u, v) \in U.$$

By regularity, we can assume that $\dfrac{\partial(x, y)}{\partial(u, v)}(q) \neq 0$. Define a map $F : U \times \mathbb{R} \to \mathbb{R}^3$ by

$$F(u, v, t) = X(u, v) + t(0, 0, 1) = (x(u, v), y(u, v), z(u, v) + t), \quad (u, v) \in U,\ t \in \mathbb{R}.$$

Then

$$\det(dF_{(q,0)}) = \frac{\partial(x, y)}{\partial(u, v)}(q) \neq 0.$$

By the inverse function theorem, there exist open neighborhoods W_1 of $(q, 0)$ and W_2 of $F(q, 0)$ such that $F : W_1 \to W_2$ is a diffeomorphism. Set $V = W_1 \cap U \subset \mathbb{R}^2$ and observe that the restriction $F|_V = X|_V$. Thus, $X(V) = F(V)$ is diffeomorphic to V, and hence a regular surface.

The Tangent Plane; The Differential of a Map

Definition Let S be a regular surface and p be a point in S. Then a vector w is called a tangent vector to S at the point p if there exists a differentiable parametrized curve $\alpha : (-\varepsilon, \varepsilon) \to S$ such that $\alpha(0) = p$ and $\alpha'(0) = w$. The set of tangent vectors to S at p, denoted by T_pS, is called the tangent plane to S at p.

Proposition 1. Let $X : U \subset \mathbb{R}^2 \to S$ be a parametrization of a regular surface S and let $q \in U$. The vector subspace of dimension 2,

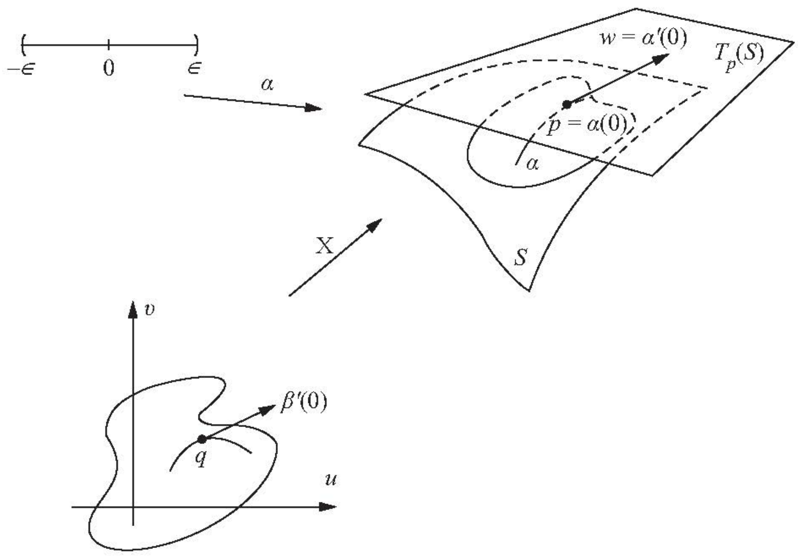
$$dX_q(\mathbb{R}^2) \subset \mathbb{R}^3,$$

coincides with the set of tangent vectors to X at $X(q)$, i.e. $dX_q(\mathbb{R}^2) = T_{X(q)}S$.

Proof Let $w \in T_{X(q)}S$ be a tangent vector at $X(q)$, that is, let $w = \alpha'(0)$, where $\alpha : (-\varepsilon, \varepsilon) \to X(U) \subset S$ is differentiable and $\alpha(0) = X(q)$. Since U and $X(U)$ are diffeomorphic, $X^{-1} : X(U) \to U$ is differentiable and the curve $\beta = X^{-1} \circ \alpha : (-\varepsilon, \varepsilon) \to U$ is differentiable. This implies that $X \circ \beta(t) = \alpha(t)$ for $t \in (-\varepsilon, \varepsilon)$ and

$$\frac{d}{dt} X \circ \beta(t)|_{t=0} = \frac{d}{dt}\alpha(t)|_{t=0} \implies dX_q(\beta'(0)) = dX_{\beta(0)}(\beta'(0)) = \alpha'(0) = w.$$

Hence, $w \in dX_q(\mathbb{R}^2) \implies T_{X(q)}S \subseteq dX_q(\mathbb{R}^2)$.



On the other hand, let $w = dX_q(v)$, where $v \in \mathbb{R}^2$. It is clear that v is the velocity vector of the curve $\gamma : (-\varepsilon, \varepsilon) \to U$ given by

$$\gamma(t) = tv + q, \quad t \in (-\varepsilon, \varepsilon).$$

By the definition of the differential, $w = \alpha'(0)$, where $\alpha = X \circ \gamma$. This shows that $w \in T_{X(q)}S$ is a tangent vector and $dX_q(\mathbb{R}^2) \subseteq T_{X(q)}S$.

By the above proposition, the plane $dX_q(\mathbb{R}^2)$, which passes through $X(q) = p$, does not depend on the parametrization X. This plane will be called the tangent plane to S at p and will be denoted by T_pS. The choice of the parametrization X determines a basis $\{(\partial X/\partial u)(q), (\partial X/\partial v)(q)\}$ of T_pS, called the basis associated to X. Sometimes it is convenient to write $\partial X/\partial u = X_u$ and $\partial X/\partial v = X_v$.

The coordinates of a vector $w \in T_pS$ in the basis associated to a parametrization X are determined as follows. Let $w = \alpha'(0)$ for some $\alpha(t) = X(u(t), v(t))$ with $(u(0), v(0)) = q = X^{-1}(p)$. Thus,

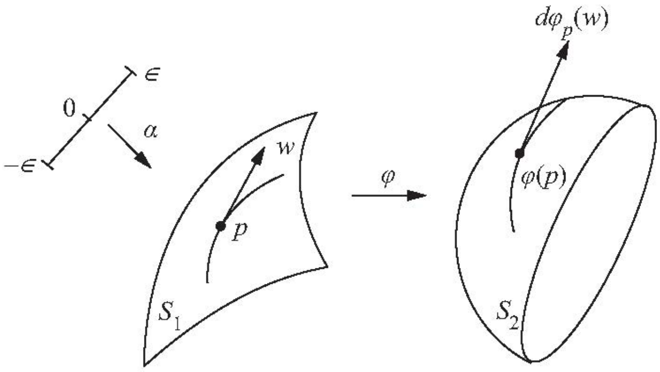
$$w = \alpha'(0) = \frac{d}{dt}X(u(t), v(t))|_{t=0} = X_u(q)u'(0) + X_v(q)v'(0).$$

That is, in the basis $\{X_u(q), X_v(q)\}$ of T_pS, w has coordinates $(u'(0), v'(0))$, where $(u(t), v(t))$ is the expression, in the parametrization X, of a curve whose velocity vector at $t = 0$ is w.

With the notion of a tangent plane, we can talk about the differential of a (differentiable) map between surfaces. Let S_1 and S_2 be two regular surfaces and let $\varphi : V \subset S_1 \to S_2$ be a differentiable mapping of an open set V of S_1 into S_2. For each $p \in V$, there is a map, called the differential of φ at p,

$$d\varphi_p : T_pS_1 \to T_{\varphi(p)}S_2$$

defined as follows.



For each $w \in T_pS_1$, let $\alpha : (-\varepsilon, \varepsilon) \to V \subset S_1$ be a differentiable parametrized curve such that $\alpha(0) = p$ and $\alpha'(0) = w$. Since $\beta = \varphi \circ \alpha : (-\varepsilon, \varepsilon) \to S_2$ is a curve in S_2 such that $\beta(0) = \varphi(p)$, this implies that $\beta'(0) \in T_{\varphi(p)}S_2$ and it is defined to be $d\varphi_p(w)$, i.e. $\beta'(0) = d\varphi_p(w)$.

Note that the coordinates of a vector $\beta'(0) \in T_{\varphi(p)}S_2$ in the basis associated to a parametrization $\bar{X}$ are determined as follows. Let $\beta(t) = \varphi \circ \alpha(t) = \bar{X}(\bar{u}(t), \bar{v}(t))$ with $(\bar{u}(0), \bar{v}(0)) = r = \bar{X}^{-1}(\varphi(p))$. Then

$$\beta'(0) = \frac{d}{dt}\bar{X}(\bar{u}(t), \bar{v}(t))|_{t=0} = \bar{X}_{\bar{u}}(r)\bar{u}'(0) + \bar{X}_{\bar{v}}(r)\bar{v}'(0).$$

That is, in the basis $\{\bar{X}_{\bar{u}}(r), \bar{X}_{\bar{v}}(r)\}$ of $T_{\varphi(p)}S_2$, $\beta'(0)$ has coordinates $(\bar{u}'(0), \bar{v}'(0))$.

Proposition 2. In the discussion above, given w, the vector $\beta'(0)$ does not depend on the choice of α. The map $d\varphi_p : T_pS_1 \to T_{\varphi(p)}S_2$ defined by $d\varphi_p(w) = \beta'(0)$ is linear.

Proof Idea: Find a matrix representation of $d\varphi_p : T_pS_1 \to T_{\varphi(p)}S_2$ in the bases $\{X_u, X_v\}$ of T_pS_1 and $\{\bar{X}_{\bar{u}}, \bar{X}_{\bar{v}}\}$ of $T_{\varphi(p)}S_2$.

Let $X(u, v)$, $\bar{X}(\bar{u}, \bar{v})$, be parametrizations in neighborhoods of $p = X(q) = \bar{X}(r)$ and $\varphi(p)$, respectively, such that $\varphi(X(u, v)) \in \bar{X}(\bar{U})$ for all $(u, v) \in U$. Let

$$\alpha(t) = X(u(t), v(t)) : (-\varepsilon, \varepsilon) \to S_1, \quad \beta(t) = \varphi \circ \alpha(t) = \bar{X}(\bar{u}(t), \bar{v}(t))$$

such that $\alpha(0) = p$ and $\beta(0) = \varphi(p)$.

Consider the map $\Phi = \bar{X}^{-1} \circ \varphi \circ X : U \to \bar{U}$ given by

$$\Phi(u, v) = (\varphi_1(u, v), \varphi_2(u, v)) \quad \text{for } (u, v) \in U,$$

and the curve $\bar{X}^{-1} \circ \beta = \bar{X}^{-1} \circ \varphi \circ \alpha = \bar{X}^{-1} \circ \varphi \circ X(u(t), v(t))$ defined by

$$(\bar{u}(t), \bar{v}(t)) = \Phi(u(t), v(t)) = (\varphi_1(u(t), v(t)), \varphi_2(u(t), v(t))) \quad \text{for } t \in (-\varepsilon, \varepsilon).$$

Since

$$\begin{pmatrix} \bar{u}'(0) \\ \bar{v}'(0) \end{pmatrix} = \begin{pmatrix} \frac{\partial \varphi_1}{\partial u}u'(0) + \frac{\partial \varphi_1}{\partial v}v'(0) \\ \frac{\partial \varphi_2}{\partial u}u'(0) + \frac{\partial \varphi_2}{\partial v}v'(0) \end{pmatrix} = \begin{pmatrix} \frac{\partial \varphi_1}{\partial u} & \frac{\partial \varphi_1}{\partial v} \\ \frac{\partial \varphi_2}{\partial u} & \frac{\partial \varphi_2}{\partial v} \end{pmatrix} \begin{pmatrix} u'(0) \\ v'(0) \end{pmatrix},$$

and since $dX_q(\mathbb{R}^2) = T_pS_1$, $d\bar{X}_p^{-1}(T_{\varphi(p)}S_2) = \mathbb{R}^2$, and

- $(u'(0), v'(0)) \in \mathbb{R}^2$ is the coordinates of w in the basis $\{X_u, X_v\}$ of T_pS_1,
- $(\bar{u}'(0), \bar{v}'(0)) \in \mathbb{R}^2$ is the coordinates of $\beta'(0)$ in the basis $\{\bar{X}_{\bar{u}}, \bar{X}_{\bar{v}}\}$ of $T_{\varphi(p)}S_2$,
- the matrix $d\Phi_q = \begin{pmatrix} \partial\varphi_1/\partial u & \partial\varphi_1/\partial v \\ \partial\varphi_2/\partial u & \partial\varphi_2/\partial v \end{pmatrix}$ depends only on Φ,

i.e.

$$\beta'(0) = d\varphi_p(w) \iff \underbrace{\begin{pmatrix} \bar{u}'(0) \\ \bar{v}'(0) \end{pmatrix}}_{\text{coordinates of } \beta'(0)} = \begin{pmatrix} \dfrac{\partial\varphi_1}{\partial u} & \dfrac{\partial\varphi_1}{\partial v} \\ \dfrac{\partial\varphi_2}{\partial u} & \dfrac{\partial\varphi_2}{\partial v} \end{pmatrix} \underbrace{\begin{pmatrix} u'(0) \\ v'(0) \end{pmatrix}}_{\text{coordinates of } w},$$

we conclude that

- $\beta'(0)$ is independent of α,
- the matrix $d\Phi_q$ is a linear map from T_pS_1 in the basis $\{X_u, X_v\}$ to $T_{\varphi(p)}S_2$ in the basis $\{\bar{X}_{\bar{u}}, \bar{X}_{\bar{v}}\}$,
- $d\Phi_q$ is the matrix representation of $d\varphi_p$ in the bases $\{X_u, X_v\}$ of T_pS_1 and $\{\bar{X}_{\bar{u}}, \bar{X}_{\bar{v}}\}$ of $T_{\varphi(p)}S_2$.

Hence $d\varphi_p$ is a linear mapping from T_pS_1 into $T_{\varphi(p)}S_2$.

The linear map $d\varphi_p$ defined by Prop. 2 is called the differential of φ at $p \in S_1$. In a similar way we define the differential of a (differentiable) function $f : U \subset S \to \mathbb{R}$ at $p \in U$ as a linear map $df_p : T_pS \to \mathbb{R}$.

Example Let $v \in \mathbb{R}^3$ be a unit vector and let $h : S \to \mathbb{R}$, $h(p) = v \cdot p$, $p \in S$, be the height function defined in Example 1 of Sec. 2-3. To compute $dh_p(w)$, $w \in T_pS$, choose a differentiable curve $\alpha : (-\varepsilon, \varepsilon) \to S$ with $\alpha(0) = p$, $\alpha'(0) = w$. Since $h(\alpha(t)) = \alpha(t) \cdot v$, we obtain

$$dh_p(w) = \frac{d}{dt} h(\alpha(t))|_{t=0} = \alpha'(0) \cdot v = w \cdot v.$$

Example Let $S^2 \subset \mathbb{R}^3$ be the unit sphere

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$$

and let $R_{z,\theta} : \mathbb{R}^3 \to \mathbb{R}^3$ be the rotation of angle θ about the z axis. Then $R_{z,\theta}$ restricted to S^2 is a differentiable map of S^2. We shall compute $(dR_{z,\theta})_p(w)$, $p \in S^2$, $w \in T_pS^2$. Let $\alpha : (-\varepsilon, \varepsilon) \to S^2$ be a differentiable curve with $\alpha(0) = p$, $\alpha'(0) = w$. Then, since $R_{z,\theta}$ is linear,

$$(dR_{z,\theta})_p(w) = \frac{d}{dt} (R_{z,\theta} \circ \alpha(t)) |_{t=0} = R_{z,\theta}(\alpha'(0)) = R_{z,\theta}(w).$$

Observe that $R_{z,\theta}$ leaves the north pole $N = (0, 0, 1)$ fixed, and that $(dR_{z,\theta})_N : T_NS^2 \to T_NS^2$ is just a rotation of angle θ in the plane T_NS^2.

We shall say that a mapping $\varphi : U \subset S_1 \to S_2$ a local diffeomorphism at $p \in U$ if there exists an open neighborhood $V \subset U$ of p such that φ restricted to V is a diffeomorphism onto an open set $\varphi(V) \subset S_2$. In these terms, the version of the inverse of function theorem for surfaces is expressed as follows.

Proposition 3. If S_1 and S_2 are regular surfaces and $\varphi : U \subset S_1 \to S_2$ is a differentiable mapping of an open set $U \subset S_1$ such that the differential $d\varphi_p$ of φ at $p \in U$ is an isomorphism, then φ is a local diffeomorphism at p.

Remark By fixing a parametrization $X : U \subset \mathbb{R}^2 \to S$ at $p \in S$, we can make a definite choice of a unit normal vector at each point $q \in X(U)$ by the rule

$$N(q) = \frac{X_u \wedge X_v}{|X_u \wedge X_v|}(q).$$

Note that $N : X(U) \to \mathbb{R}^3$ is a differentiable map on $X(U) \subset S$ and it is not always possible to extend this map differentiably to the whole surface S.

The First Fundamental Form; Area

Definition Let S be a regular surface in $\mathbb{R}^3$. For each $p \in S$ and tangent vectors w_1, $w_2 \in T_pS \subset \mathbb{R}^3$, there is an (induced) inner product

$$\langle\ ,\ \rangle_p : T_pS \times T_pS \to \mathbb{R}$$

defined by

$$\langle w_1, w_2\rangle_p = \langle w_1, w_2\rangle = \text{the inner product of } w_1 \text{ and } w_2 \text{ as vectors in } \mathbb{R}^3.$$

To this inner product, which is a symmetric bilinear form, there corresponds a quadratic form

$$I_p : T_pS \to \mathbb{R}$$

defined by

$$I_p(w) = \langle w, w\rangle_p = \langle w, w\rangle = |w|^2 \geq 0 \quad \text{for } w \in T_pS,$$

and the quadratic form I_p on T_pS is called the first fundamental form of the regular surface $S \subset \mathbb{R}^3$ at $p \in S$.

Remark Let U be an open set in the uv-plane and let $X : U \subset \mathbb{R}^2 \to S$ be a parametrization of the regular surface at $p = X(u_0, v_0) \in S$. For each $w \in T_pS$, since there is a parametrized curve

$$\alpha(t) = X(u(t), v(t)) \in X(U),\ t \in (-\varepsilon, \varepsilon),\ \text{such that } p = \alpha(0) = X(u_0, v_0) \text{ and } w = \alpha'(0),$$

we have

$$\begin{aligned} I_p(w) = I_p(\alpha'(0)) &= \langle \alpha'(0), \alpha'(0)\rangle_p \\ &= \langle X_u u' + X_v v', X_u u' + X_v v'\rangle_p \\ &= \langle X_u, X_u\rangle_p (u')^2 + 2\langle X_u, X_v\rangle_p u'\, v' + \langle X_v, X_v\rangle_p (v')^2 \\ &= E(u')^2 + 2F u'\, v' + G(v')^2, \end{aligned}$$

where the values of the functions involved are computed for $t = 0$, and

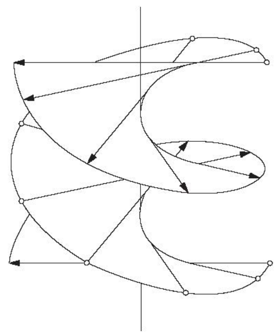
$$\begin{aligned} E(u_0, v_0) &= \langle X_u, X_u\rangle_p, \\ F(u_0, v_0) &= \langle X_u, X_v\rangle_p, \\ G(u_0, v_0) &= \langle X_v, X_v\rangle_p \end{aligned}$$

are the coefficients of the first fundamental form in the basis $\{X_u, X_v\}$ of T_pS. By letting p run in the coordinate neighborhood corresponding to $X(u,v)$ we obtain functions $E(u,v)$, $F(u,v)$, $G(u,v)$ which are differentiable in that neighborhood.

From now on we shall drop the subsript p in the indication of the inner product $\langle\, ,\, \rangle_p$, or the quadratic form I_p when it is clear from the context which point we are referring to. It will also be convenient to denote the natural inner product of $\mathbb{R}^3$ by the same symbol $\langle\, ,\, \rangle$ rather than the previous dot.

Examples

1. Let $p_0 = (x_0, y_0, z_0)$ be a point in $\mathbb{R}^3$, $w_1 = (a_1, a_2, a_3)$ and $w_2 = (b_1, b_2, b_3)$ be orthonormal vectors in $\mathbb{R}^3$ and
$$P = \{X(u,v) = p_0 + uw_1 + vw_2 \mid (u,v) \in \mathbb{R}^2\}.$$
Then P is a plane and $E = 1$, $F = 0$ and $G = 1$ at every point in P.
2. Let $U = \{(u,v) \mid 0 < u < 2\pi,\ -\infty < v < \infty\}$ and
$$X(u,v) = (\cos u, \sin u, v), \quad \text{for } (u,v) \in U.$$
Then $X(U)$ is an open subset of the cylinder $C = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$ and $E = 1$, $F = 0$ and $G = 1$ at every point $X(u,v)$ in C.
We remark that, although the cylinder and the plane are distinct surfaces, we obtain the same result in both cases.
3. Let $a > 0$ and $\alpha(u) = (\cos u, \sin u, au)$ denote a helix. Through each point of the helix, draw a line parallel to the xy plane and intersecting the z axis. The surface generated by these lines is called a helicoid and admits the following parametrization
$$X(u,v) = (v\cos u, v\sin u, au), \quad (u,v) \in U = \{(u,v) \mid 0 < u < 2\pi,\ -\infty < v < \infty\}.$$
Then $E(u,v) = v^2 + a^2$, $F(u,v) = 0$ and $G(u,v) = 1$.



Remarks

1. Let S be a regular surface S in $\mathbb{R}^3$ and let the arc length $s = s(t)$ of a parametrized curve $\alpha : I \to S$ be given by
$$s(t) = \int_0^t |\alpha'(\tau)|\, d\tau = \int_0^t \sqrt{I(\alpha'(\tau))}\, d\tau.$$
In particular, if $\alpha(t) = X(u(t), v(t))$ is contained in a coordinate neighborhood corresponding to the parametrization $X(u,v)$, we can compute the arc length of α between, say, 0 and t by
$$s(t) = \int_0^t \sqrt{E(u')^2 + 2Fu'v' + G(v')^2}\, d\tau$$

which implies that

$$\left(\frac{ds}{dt}\right)^2 = E\left(\frac{du}{dt}\right)^2 + 2F\frac{du}{dt}\frac{dv}{dt} + G\left(\frac{dv}{dt}\right)^2$$

and the "element" of arc length, ds of S, satisfies that

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2.$$

2. The angle θ under which two parametrized regular curves $\alpha : I \to S$, $\beta : I \to S$ intersect at $t = t_0$ is given by
$$\cos\theta = \frac{\langle \alpha'(t_0), \beta'(t_0)\rangle}{|\alpha'(t_0)||\beta'(t_0)|}.$$
In particular, the angle φ of the coordinate curves of a parametrization $X(u,v)$ is
$$\cos\varphi = \frac{\langle X_u, X_v\rangle}{|X_u||X_v|} = \frac{F}{\sqrt{EG}}$$
it follows that the coordinate curves of a parametrization are orthogonal if and only if $F(u,v) = 0$ for all (u,v). Such a parametrization is called an orthogonal parametrization.

Definition Let $R \subset S$ be a bounded region of a regular surface contained in the coordinate neighborhood of the parametrization $X : U \subset \mathbb{R}^2 \to S$. Then the area $A(R)$ of $R = X(Q)$ is given by
$$A(R) = \iint_Q |X_u \wedge X_v|\, du\, dv, \quad \text{where } Q = X^{-1}(R).$$

Claim The integral $\displaystyle\iint_Q |X_u \wedge X_v|\, du\, dv$ does not depend on the parametrization X.

Proof of the Claim Suppose that $\bar{U}$ is an open set in the $\bar{u}\bar{v}$-plane, $\bar{X} : \bar{U} \subset \mathbb{R}^2 \to S$ is another parametrization such that $R \subset \bar{X}(\bar{U})$, and $\bar{Q} = \bar{X}^{-1}(R)$, $(\bar{u}, \bar{v}) \in \bar{Q}$, then

$$\bar{X}_{\bar{u}} = X_u\frac{\partial u}{\partial \bar{u}} + X_v\frac{\partial v}{\partial \bar{u}}, \quad \bar{X}_{\bar{v}} = X_u\frac{\partial u}{\partial \bar{v}} + X_v\frac{\partial v}{\partial \bar{v}} \implies |\bar{X}_{\bar{u}} \wedge \bar{X}_{\bar{v}}| = |X_u \wedge X_v| \left|\frac{\partial(u,v)}{\partial(\bar{u},\bar{v})}\right|,$$

and
$$\iint_{\bar{Q}} |\bar{X}_{\bar{u}} \wedge \bar{X}_{\bar{v}}|\, d\bar{u}\, d\bar{v} = \iint_{\bar{Q}} |X_u \wedge X_v| \left|\frac{\partial(u,v)}{\partial(\bar{u},\bar{v})}\right| d\bar{u}\, d\bar{v} = \iint_Q |X_u \wedge X_v|\, du\, dv.$$
Thus the definition of the area of R does not depend on the parametrization X.

Remark Since

$$|X_u \wedge X_v|^2 + \langle X_u, X_v\rangle^2 = |X_u|^2\,|X_v|^2(\sin^2\theta + \cos^2\theta) = |X_u|^2\,|X_v|^2 \implies |X_u \wedge X_v|^2 = EG - F^2,$$

the area of $R = X(Q) \subset S$ can be written as

$$A(R) = \iint_Q |X_u \wedge X_v|\, du\, dv = \iint_Q \sqrt{EG - F^2}\, du\, dv.$$

Example Let $a > r > 0$, $S^1 = \{(y,z) \mid (y-a)^2 + z^2 = r^2\}$ and T be the torus generated by rotating S^1 about z-axis. Find the area of T.

Consider the coordinate neighborhood corresponding to the parametrization

$$X(u,v) = ((r\cos u + a)\cos v, (r\cos u + a)\sin v, r\sin u) \quad \text{where } 0 < u < 2\pi,\ 0 < v < 2\pi$$

which covers the torus, except for a meridian and a parallel. The coefficients of the first fundamental form are

$$E = r^2, \quad F = 0, \quad G = (r\cos u + a)^2 \implies \sqrt{EG - F^2} = r(r\cos u + a).$$

Now, consider the region R_ε obtained as the image by X of the region Q_ε, $\varepsilon > 0$ and small, given by

$$Q_\varepsilon = \{(u,v) \in \mathbb{R}^2 \mid 0 < \varepsilon \le u \le 2\pi - \varepsilon,\, 0 < \varepsilon \le v \le 2\pi - \varepsilon\}.$$

Then

$$A(T) = \lim_{\varepsilon\to 0} A(R_\varepsilon) = \lim_{\varepsilon\to 0} \iint_{Q_\varepsilon} r(r\cos u + a)\, du\, dv = \lim_{\varepsilon\to 0} \int_\varepsilon^{2\pi-\varepsilon} (r^2\cos u + ra)\, du \int_\varepsilon^{2\pi-\varepsilon} dv = 4\pi^2 ra.$$

Orientation of Surfaces

Let $X(u,v)$ be a parametrization of a neighborhood of a point p of a regular surface S, we determine an orientation of the tangent plane T_pS, namely, the orientation of the associated ordered basis $\{X_u, X_v\}$. If p belongs to the coordinate neighborhood of another parametrization $\bar{X}(\bar{u},\bar{v})$, the new basis $\{\bar{X}_{\bar{u}}, \bar{X}_{\bar{v}}\}$ is expressed in terms of the first one by

$$\begin{aligned}\bar{X}_{\bar{u}} &= X_u \frac{\partial u}{\partial \bar{u}} + X_v \frac{\partial v}{\partial \bar{u}},\\ \bar{X}_{\bar{v}} &= X_u \frac{\partial u}{\partial \bar{v}} + X_v \frac{\partial v}{\partial \bar{v}},\end{aligned}$$

where $u = u(\bar{u},\bar{v})$ and $v = v(\bar{u},\bar{v})$ are the expressions of the change of coordinates. The bases $\{X_u, X_v\}$ and $\{\bar{X}_{\bar{u}}, \bar{X}_{\bar{v}}\}$ determine, therefore, the same orientation of T_pS if and only if the Jacobian $\dfrac{\partial(u,v)}{\partial(\bar{u},\bar{v})}$ of the coordinate change is positive.

Definition A regular surface S is called orientable if it is possible to cover it with a family of coordinate neighborhoods in such a way that if a point $p \in S$ belongs to two neighborhoods of this family, then the change of coordinates has positive Jacobian at p. The choice of such a family is called an orientation of S, and S, in this case, is called oriented. If such a choice is not possible, the surface is called nonorientable.

Example A surface which is the graph of a differentiable function is an orientable surface. In fact, all surfaces which can be covered by one coordinate neighborhood are trivially orientable.

Given a system of coordinates $X(u,v)$ at p, we have a definite choice of a unit normal vector N at p by the rule

$$N(p) = \frac{X_u \wedge X_v}{|X_u \wedge X_v|}(p).$$

Taking another system of local coordinates $\bar{X}(\bar{u},\bar{v})$ at p, we see that

$$\bar{X}_{\bar{u}} \wedge \bar{X}_{\bar{v}} = (X_u \wedge X_v)\,\frac{\partial(u,v)}{\partial(\bar{u},\bar{v})},$$

where $\dfrac{\partial(u,v)}{\partial(\bar{u},\bar{v})}$ is the Jacobian of the coordinate change. Hence, N will preserve its sign or change it, depending on whether $\dfrac{\partial(u,v)}{\partial(\bar{u},\bar{v})}$ is positive or negative, respectively.

Definition Let U be an open subset of a regular surface S. A map $N : U \to \mathbb{R}^3$ is called a differentiable field of unit normal vectors on an open set $U \subset S$ if $N : U \to \mathbb{R}^3$ is differentiable with $|N(q)| = 1$ at each $q \in U$.

Proposition A regular surface $S \subset \mathbb{R}^3$ is orientable if and only if there exists a differentiable field of unit normal vectors $N : S \to \mathbb{R}^3$ on S.

Outline of the Proof If S is orientable, it is possible to cover it with a family of coordinate neighborhoods so that, in the intersection of any two of them, the change of coordinates has a positive Jacobian. For each $p \in S$, let $N : S \to \mathbb{R}^3$ be defined by

$$N(p) = \frac{X_u \wedge X_v}{|X_u \wedge X_v|}(p),$$

where $X(u,v)$ is a parametrization of S at p. Since S is orientable, N is well-defined and is a differentiable field of unit normal vectors on S.

Conversely, suppose that there exists a differentiable field of unit normal vectors $N : S \to \mathbb{R}^3$ on S. For each $p \in S$, let $X(u,v)$ be a parametrization of S at p such that

$$f(p) = \langle N(p), \frac{X_u \wedge X_v}{|X_u \wedge X_v|}\rangle(p) = 1.$$

Since the collection of all such coordinate neighborhoods

$$\{X : U \subset \mathbb{R}^2 \to S \mid S = \cup X(U), \text{ and } f(q) = 1 \text{ for each } q \in X(U)\}$$

covers S and the change of coordinates has positive Jacobian at each $p \in S$, the regular surface S is orientable.

Proposition If a regular surface is given by $S = \{(x,y,z) \in \mathbb{R}^3 \mid f(x,y,z) = a\}$, where $f : U \subset \mathbb{R}^3 \to \mathbb{R}$ is differentiable and a is a regular value of f, then S is orientable.

Outline of the Proof For each $p = (x,y,z) \in S$, let $N(p)$ be the unit normal vector defined by

$$N(p) = \frac{\nabla f}{|\nabla f|}(p),$$

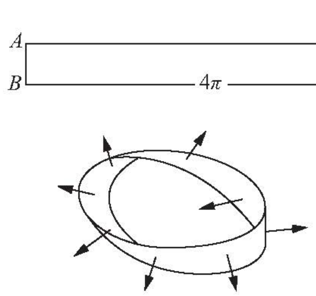
where $\nabla f(p) = (f_x, f_y, f_z)(p)$ is the gradient vector of f at p. Since N is differentiable, S is orientable.

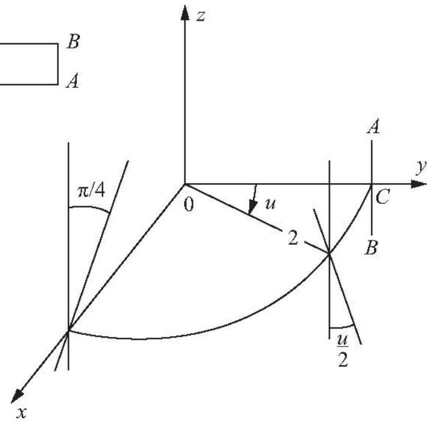
Example The sphere is an orientable surface.

Example Let S^1 be the circle given by $x^2 + y^2 = 4$ and AB be the open segment given in the yz plane by $y = 2$, $|z| < 1$. Move the center C of AB along S^1 and turn AB about C in the Cz plane in such a manner that when C has passed through an angle u, AB has rotated by an angle $u/2$. When C completes one trip around the circle, AB returns to its initial position, with its end points inverted and we obtain a nonorientable surface M, called the Möbius strip.

Let $U = \{(u,v) \mid 0 < u < 2\pi, \ -1 < v < 1\}$ and the coordinates X, $\bar{X} : U \to M$ be defined by

$$\begin{aligned} X(u,v) &= \left(\left(2 - v\sin\frac{u}{2}\right)\sin u, \left(2 - v\sin\frac{u}{2}\right)\cos u, \, v\cos\frac{u}{2}\right), \quad (u,v) \in U \\ \bar{X}(\bar{u},\bar{v}) &= \left(\left[2 - \bar{v}\sin\left(\frac{\pi}{4} + \frac{\bar{u}}{2}\right)\right]\cos\bar{u}, \, -\left[2 - \bar{v}\sin\left(\frac{\pi}{4} + \frac{\bar{u}}{2}\right)\right]\sin\bar{u}, \, \bar{v}\cos\left(\frac{\pi}{4} + \frac{\bar{u}}{2}\right)\right), \quad (\bar{u},\bar{v}) \in U \end{aligned}$$





Observe that the intersection of the two coordinate neighborhoods is not connected but consists of two connected components

$$W_1 = \{X(u,v) \mid \frac{\pi}{2} < u < 2\pi\}, \quad W_2 = \{X(u,v) \mid 0 < u < \frac{\pi}{2}\}.$$

The change of coordinates is given by

$$\begin{aligned} \bar{u} &= u - \frac{\pi}{2}, \quad \bar{v} = \quad v \quad \text{in } W_1, \\ \bar{u} &= u + \frac{3\pi}{2}, \quad \bar{v} = -v \quad \text{in } W_2. \end{aligned}$$

It follows that

$$\begin{aligned} \frac{\partial(\bar{u},\bar{v})}{\partial(u,v)} &= \quad 1 > 0 \quad \text{in } W_1, \\ \frac{\partial(\bar{u},\bar{v})}{\partial(u,v)} &= -1 < 0 \quad \text{in } W_2. \end{aligned}$$

Suppose that it is possible to define a differentiable field of unit normal vectors $N : M \to \mathbb{R}^3$. Interchanging u and v if necessary, we can assume that

$$N(p) = \frac{X_u \wedge X_v}{|X_u \wedge X_v|}(p)$$

for any p in the coordinate neighborhood of $X(u,v)$. Analogously, we may assume that

$$N(p) = \frac{\bar{X}_{\bar{u}} \wedge \bar{X}_{\bar{v}}}{|\bar{X}_{\bar{u}} \wedge \bar{X}_{\bar{v}}|}$$

at all points of the coordinate neighborhood of $\bar{X}(\bar{u},\bar{v})$. However, the Jacobian of the change of coordinates must be -1 in either W_1 or W_2 (depending on what changes of the type $u \to v$, $\bar{u} \to \bar{v}$ has to be made). If p is a point of that component of the intersection, then $N(p) = -N(p)$, which is a contradiction.